Extended Jaynes–Cummings models and (quasi)-exact solvability

2006 J. Phys. A: Math. Gen. 39 9817

(http://iopscience.iop.org/0305-4470/39/31/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **39** (2006) 9817–9830 doi:10.1088/0305-4470/39/31/011

Extended Jaynes–Cummings models and (quasi)-exact solvability

Yves Brihaye and Ancilla Nininahazwe

Faculté des Sciences, Université de Mons-Hainaut, B-7000 Mons, Belgium

E-mail: brihaye@umh.ac.be and nininaha@yahoo.fr

Received 16 March 2006, in final form 6 June 2006
Published 19 July 2006
Online at stacks.iop.org/JPhysA/39/9817

Abstract

The original Jaynes–Cummings model is described by a Hamiltonian which is Hermitian and exactly solvable. Here, we extend this model by several types of interactions leading to a non-Hermitian operator which does not satisfy the physical condition of spacetime reflection symmetry (PT symmetry). The new Hamiltonians are either exactly solvable admitting an entirely real spectrum or quasi-exactly solvable with a real algebraic part of their spectrum.

PACS numbers: 03.65.Pm, 31.30.Jv

1. Introduction

Several new theoretical aspects of quantum mechanics have been developed in the last few years. In a series of papers (see, e.g., [1, 2] and [3] for a recent review), it is shown that the traditional self-adjointness requirement of the Hamiltonian operator is not a necessary condition to guarantee a real spectrum and that the weaker condition of PT invariance of the Hamiltonian is sufficient for that purpose. An alternative possibility for an operator to admit a real spectrum is also developed in [4]. It is the notion of pseudo-Hermiticity. Following the ideas of [4], let us recall that a Hamiltonian is called η pseudo-Hermitian if it satisfies the relation $\eta H \eta^{-1} = H^{\dagger}$, where η denotes a linear Hermitian operator. It is this new notion (i.e., pseudo-Hermiticity property) of non-Hermitian Hamiltonians which explains the reality of their energy spectrum. This important property has further been considered in [5, 6].

Another direction of development of quantum mechanics is the notion of quasi-exact solvability [7, 8]. It provides techniques to construct linear operators preserving a *finite-dimensional* subspace $\mathcal{V}$ of the Hilbert space. Accordingly, the so-called quasi-exactly solvable (QES) operators, once restricted on $\mathcal{V}$, can be diagonalized by means of algebraic methods. The QES property is strongly connected to finite-dimensional representation of Lie or graded Lie algebras [7, 9, 10]. Amongst many models used to describe quantum properties of physical systems, the Jaynes–Cummings model plays an important role [11–14]. It describes in a simple way the interaction of photons with a spin-$\frac{1}{2}$ particle. From the mathematical point of view, the Jaynes–Cummings model is described by a self-adjoint operator and it is completely solvable in a sense that the entire spectrum can be computed algebraically. The purpose of this paper is to consider operators generalizing the Jaynes–Cummings Hamiltonians which are neither self-adjoint nor PT invariant but are pseudo-Hermitian with respect to two different operators. In particular, from the original Jaynes–Cummings model (JCM in the following), we construct an extended one by adding a polynomial of the form $P(a^\dagger, a)$ ($a^\dagger, a$ are the usual creation and annihilation operators) of degree $d \geqslant 2$ in the diagonal part of the Hamiltonian. Some particular choices of P are constructed in such a way that the resulting operator becomes QES. The non-diagonal interaction part is also modified in such a way that (i) multiple photon exchanges are allowed and (ii) the full operator can be Hermitian or pseudo-Hermitian.

In section 2, we give the Hamiltonian considered in [5] and express it in terms of differential operator of a real variable x. This reveals the exact solvability of the Hamiltonian if these differential operators preserve one (or more) set of polynomials of appropriate degrees in x. In section 3, we propose a family of operators which generalize the original JC Hamiltonian in several respects. The (pseudo)-Hermiticity of these operators are analysed and the spectra and the eigenvectors are computed in details for a number of them. The differences in the spectrum corresponding to Hermitian and pseudo-Hermitian are pointed out. In particular, the energy eigenvalues are entirely real in spite of the fact that they are associated with a non-Hermitian and non PT-invariant Hamiltonian. The reality of those eigenvalues is ensured by the pseudo-Hermiticity of the Hamiltonians. Section 4 is devoted to QES extensions of the JCM. These are constructed in such a way that, both, one-photon and two-photon exchange terms coexist in the non-diagonal interacting terms. By construction, these new models preserve finite-dimensional vector spaces of the Hilbert spaces. The algebraic part of the spectrum is computed in section 5. Further properties of these new types of QES operators, say H_T, can be discussed. Namely, following the ideas of [15] we show in section 6 that the solutions of the spectral equation $H_T\psi = E\psi$ for generic values of E lead to new types of recurrence relations. The relations between H_T and specific graded algebras are pointed out in section 7. Finally, section 8 contains our concluding remarks.

2. Exactly solvable pseudo-Hermitian Hamiltonian

In this section, we consider the Hamiltonian describing a system of a spin-$\frac{1}{2}$ particle in the external magnetic field, $\vec{B}$, which couples to a harmonic oscillator through some non-Hermitian interaction [5]:

$$H = \mu\vec{\sigma}\cdot\vec{B} + \hbar\omega a^\dagger a + \rho(\sigma_+ a - \sigma_- a^\dagger). \tag{1}$$

Here, $\vec{\sigma}$ denotes the Pauli matrices, ρ is an arbitrary real parameter and $\sigma_\pm \equiv \frac{1}{2}[\sigma_x \pm \mathrm{i}\sigma_y]$. σ_+ and σ_- can be expressed in matrix form as

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Following the ideas of [7], our purpose is to transform the above Hamiltonian into an appropriate differential operator which preserves a family of vector spaces formed by couples of polynomials in the variable x. For this purpose, we use the usual creation and annihilation operators $a^\dagger$ and a, respectively, which are defined as follows:

$$a^\dagger = \frac{p + \mathrm{i}m\omega x}{\sqrt{2m\omega\hbar}}, \qquad a = \frac{p - \mathrm{i}m\omega x}{\sqrt{2m\omega\hbar}}, \tag{3}$$

where $p = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$. The external magnetic field is chosen in z-direction (i.e., $\vec{B} = B_0\vec{z}$) so that the Hamiltonian defined in equation (1) is reduced and is given by

$$H = \frac{\epsilon}{2}\sigma_z + \hbar\omega a^\dagger a + \rho(\sigma_+ a - \sigma_- a^\dagger), \tag{4}$$

where $\epsilon = 2\mu B_0$. As $\sigma_\pm^\dagger = \sigma_\mp$, it is pointed out that this Hamiltonian is not Hermitian:

$$\begin{aligned} H^\dagger &= \frac{\epsilon}{2}\sigma_z + \hbar\omega a^\dagger a - \rho(\sigma_+ a - \sigma_- a^\dagger), \\ &\neq H. \end{aligned} \tag{5}$$

Thus as,

$$\begin{aligned} PTH(PT)^{-1} &= -\frac{\epsilon}{2}\sigma_z + \hbar\omega a^\dagger a + \rho(\sigma_+ a^\dagger - \sigma_- a), \\ &\neq H, \end{aligned} \tag{6}$$

one can easily check that the Hamiltonian (1) is not PT symmetric, i.e. $H \neq H^PT$ [1].

The next step is to write H in terms of differential operators (i.e., $p = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$) and of variable x. The purpose of these transformations is to reveal the exact solvability of the operator H by using the quasi-exactly solvable (QES) technique as has been considered in [14]. Replacing the operators $a^\dagger$ and a by their expressions (as given in equation (3)) in equation (4), the Hamiltonian of the model is written now as follows:

$$H = \frac{\epsilon}{2}\sigma_z + \frac{p^2 - m\omega + m^2\omega^2x^2}{2m} + \rho\frac{[\sigma_+(p - \mathrm{i}m\omega x) - \sigma_-(p + \mathrm{i}m\omega x)]}{\sqrt{2m\omega\hbar}}. \tag{7}$$

In order to reveal the solvability of the above operator H, we first perform the standard (often called 'gauge') transformation:

$$\tilde{H} = R^{-1}HR, \qquad R = \exp\left(-\frac{m\omega x^2}{2}\right). \tag{8}$$

After some algebra, the new Hamiltonian $\tilde{H}$ is of the form

$$\begin{aligned} \tilde{H} &= \frac{\epsilon}{2}\sigma_z - \frac{1}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \omega x\frac{\mathrm{d}}{\mathrm{d}x} + \rho\frac{[\sigma_+ p - \sigma_-(p + 2\mathrm{i}m\omega x)]}{\sqrt{2m\omega\hbar}} \\ &= \frac{\epsilon}{2}\sigma_z + \frac{p^2}{2m} + \omega x\frac{\mathrm{d}}{\mathrm{d}x} + \rho\frac{[\sigma_+ p - \sigma_-(p + 2\mathrm{i}m\omega x)]}{\sqrt{2m\omega\hbar}}. \end{aligned} \tag{9}$$

Replacing σ_z, σ_+ and σ_- by their matrix form, the final form of the Hamiltonian $\tilde{H}$ is

$$\begin{aligned} \tilde{H} &= \frac{\epsilon}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} \frac{p^2}{2m} + \omega x\frac{\mathrm{d}}{\mathrm{d}x} & 0 \\ 0 & \frac{p^2}{2m} + \omega x\frac{\mathrm{d}}{\mathrm{d}x} \end{pmatrix} - \rho\begin{pmatrix} 0 & -\frac{p}{\sqrt{2m\omega\hbar}} \\ \frac{p+2\mathrm{i}m\omega x}{\sqrt{2m\omega\hbar}} & -0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{p^2}{2m} + \omega x\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\epsilon}{2} & \rho\frac{p}{\sqrt{2m\omega\hbar}} \\ -\rho\frac{p+2\mathrm{i}m\omega x}{\sqrt{2m\omega\hbar}} & \frac{p^2}{2m} + \omega x\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\epsilon}{2} \end{pmatrix}. \end{aligned} \tag{10}$$

Hence, the operator $\tilde{H}$ is typically QES because it preserves a finite-dimensional vector spaces of polynomials, namely, $\mathcal{V}_n = (P_{n-1}(x), P_n(x))^t$ with $n \in \mathbb{N}$. Moreover, $\tilde{H}$ is exactly solvable because n does not have to be fixed (it can be any nonnegative integer).

Note that the above Hamiltonian $\tilde{H}$ is not invariant under simultaneous parity operator (P) and time reversal (T) reflection (i.e., respectively $x \to -x$ and $i \to -i$) [1]. Even if the operator $\tilde{H}$ (therefore H) is non-Hermitian and not PT invariant, it was pointed out that its spectrum is real. The reality of the eigenvalues of H is a consequence of the unbroken $P\sigma_z$ (i.e., combined parity operator P and Pauli matrices σ_z) invariance of H (i.e., $[H, P\sigma_z] = 0$).

In other words, the spectrum is real because H is pseudo-Hermitian with respect to σ_z (i.e., $\sigma_z H\sigma_z^{-1} = H^\dagger$) and also to the parity operator P (i.e., $PHP^{-1} = H^\dagger$) [4–6]. We would like to mention that it is not necessary to calculate the energy eigenvalues and their corresponding eigenvectors of H because they have been determined in [5]. In the following section, we will construct the spectrum of the generalized Hamiltonian presented in equation (1).

3. Family of exactly solvable Hamiltonians

The original JCM is defined by the Hamiltonian

$$H = \frac{\epsilon}{2}\sigma_3 + \hbar\omega a^\dagger a + \rho(a\sigma_+ + a^\dagger\sigma_-), \tag{11}$$

where ρ is a real parameter (i.e., it is a real coupling constant). It can be easily checked that the Hamiltonian H is Hermitian.

In the next section, we will consider an extension of the above JCM Hamiltonian in the form

$$H = \frac{\epsilon}{2}\sigma_3 + \hbar\omega a^\dagger a + P(a^\dagger a) + \rho(a^k\sigma_+ + \phi(a^\dagger)^k\sigma_-), \tag{12}$$

where $\phi = \pm 1$ and $P(a^\dagger a)$ denotes a polynomial of degree $d \geqslant 2, k$ is an integer $\geqslant 1$ and ρ is an arbitrary real parameter. In fact, the above Hamiltonian is non-Hermitian (i.e., for $\phi = -1$) and not PT invariant but it satisfies the pseudo-Hermiticity with the operators P (operator of parity) and σ_3 (Pauli matrices), but for $\phi = +1$ the Hamiltonian given by equation (12) becomes Hermitian. For both cases, it can be easily observed that the energy spectrum is entirely real. Thus, note that the above Hamiltonian (12) is a generalization of the Hamiltonians given by equations (1) and (11). The matrix form of H is

$$\begin{pmatrix} \hbar\omega a^\dagger a + P(a^\dagger a) + \frac{\epsilon}{2} & \rho a^k \\ \phi\rho(a^\dagger)^k & \hbar\omega a^\dagger a + P(a^\dagger a) - \frac{\epsilon}{2} \end{pmatrix} \tag{13}$$

which can be easily checked to preserve the vector spaces:

$$\mathcal{V}_n = \text{span}\left\{\begin{pmatrix} |n\rangle \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ |n+k\rangle \end{pmatrix}\right\}, \qquad n \in \mathbb{N}. \tag{14}$$

It means that the action of the operator H on the vectors states $\binom{|n\rangle}{0}$ and $\binom{0}{|n+k\rangle}$ can be expressed as linear combinations of these same states. Here, we are allowed to conclude that H is exactly solvable because it preserves the vector space $\mathcal{V}_n$ for any integer n.

The next step is to find the energy eigenvalues and their corresponding eigenvectors of the Hamiltonian H for $\phi = -1$ and for $\phi = +1$. For this purpose, we recall the following identities [5]:

$$\begin{gathered} a^\dagger a\left|n, \tfrac{1}{2}m_s\right\rangle = n\left|n, \tfrac{1}{2}m_s\right\rangle, \qquad \sigma_3\left|n, \tfrac{1}{2}m_s\right\rangle = m_s\left|n, \tfrac{1}{2}m_s\right\rangle, \qquad \sigma_+\left|n, \tfrac{1}{2}\right\rangle = 0; \\ \sigma_+\left|n, -\tfrac{1}{2}\right\rangle = \left|n, \tfrac{1}{2}\right\rangle, \qquad \sigma_-\left|n, -\tfrac{1}{2}\right\rangle = 0; \qquad \sigma_-\left|n, \tfrac{1}{2}\right\rangle = \left|n, -\tfrac{1}{2}\right\rangle, \end{gathered} \tag{15}$$

with n and $m_s = \pm 1$ are respectively the eigenvalues of the number operator $a^\dagger a$ and the operator σ_3. It is readily seen that the state $\left|0, -\frac{1}{2}\right\rangle$ is a ground state of the operator H (i.e., it is constructed by the lowest values of n and m_s which are respectively 0 and -1). We have now to consider the action of H to the state $\left|0, -\frac{1}{2}\right\rangle$ in order to find its associated eigenvalue:

$$\begin{aligned} H\left|0,-\frac{1}{2}\right\rangle &= \frac{\epsilon}{2}\sigma_3\left|0,-\frac{1}{2}\right\rangle + \hbar\omega a^\dagger a\left|0,-\frac{1}{2}\right\rangle + P(a^\dagger a)\left|0,-\frac{1}{2}\right\rangle \\ &\quad + \rho a^k\sigma_+\left|0,-\frac{1}{2}\right\rangle + \phi\rho(a^\dagger)^k\sigma_-\left|0,-\frac{1}{2}\right\rangle, \\ &= \frac{\epsilon}{2}\sigma_3\left|0,-\frac{1}{2}\right\rangle, \\ &= -\frac{\epsilon}{2}\left|0,-\frac{1}{2}\right\rangle. \end{aligned} \tag{16}$$

It is proved now that $-\frac{\epsilon}{2}$ is the eigenvalue of the ground state $\left|0,-\frac{1}{2}\right\rangle$. It is easily shown that the action of the Hamiltonian H to the next state $\left|0,\frac{1}{2}\right\rangle$ gives the following linear combination of two states and $\left|k,-\frac{1}{2}\right\rangle$:

$$H\left|0,\frac{1}{2}\right\rangle = \frac{\epsilon}{2}\left|0,\frac{1}{2}\right\rangle \pm \rho\sqrt{k!}\left|k,-\frac{1}{2}\right\rangle. \tag{17}$$

The state $\left|k,-\frac{1}{2}\right\rangle$ under the action of H leads also to a linear combination of the two above states:

$$H\left|k,-\frac{1}{2}\right\rangle = \left(\hbar\omega k + P(k) - \frac{\epsilon}{2}\right)\left|k,-\frac{1}{2}\right\rangle + \rho\sqrt{k!}\left|0,\frac{1}{2}\right\rangle. \tag{18}$$

The excited states $\left|0,\frac{1}{2}\right\rangle$ and $\left|k,-\frac{1}{2}\right\rangle$ span an invariant subspace of the space of states so that one can deduce the following Hamiltonian matrix:

$$H_k = \begin{pmatrix} \frac{\epsilon}{2} & \rho\sqrt{k!} \\ \phi\rho\sqrt{k!} & \hbar\omega k + P(k) - \frac{\epsilon}{2} \end{pmatrix}. \tag{19}$$

Particularly, for $k = 1$, $P(k) = 0$ (i.e., $P(k) = k^d, d \geqslant 2$) and considering $\phi = -1$, H_k becomes the matrix H_1 constructed in [5]. One can find the eigenvalues of the Hamiltonian matrix (19) by solving the following usual equation (i.e., characteristic polynomial equation):

$$\begin{aligned} &\det(H_k - \lambda\mathbb{1}) = 0, \\ &\begin{pmatrix} \frac{\epsilon}{2} - \lambda & \rho\sqrt{k!} \\ \phi\rho\sqrt{k!} & \hbar\omega k + P(k) - \frac{\epsilon}{2} - \lambda \end{pmatrix} = 0, \\ &4\lambda^2 - 4(\hbar\omega k + P(k))\lambda + 2(\hbar\omega k + P(k))\epsilon - \epsilon^2 + \phi 4k!\rho^2 = 0. \end{aligned} \tag{20}$$

After some algebra, the energy eigenvalues (i.e., square roots of the above equation in λ) of H_k are obtained:

$$\begin{aligned} \lambda_k^{\mathrm{I}} &= \frac{\hbar\omega k + P(k) + \sqrt{(\hbar\omega k + P(k) - \epsilon)^2 + \phi 4k!\rho^2}}{2}, \\ \lambda_k^{\mathrm{II}} &= \frac{\hbar\omega k + P(k) - \sqrt{(\hbar\omega k + P(k) - \epsilon)^2 + \phi 4k!\rho^2}}{2}. \end{aligned} \tag{21}$$

It is easily checked that for $k = 1$, $P(k) = 0$ and for $\phi = -1$, one obtains the eigenvalues $\lambda_1^{\mathrm{I,II}}$ determined in [5]. These are the energy eigenvalues of the Hamiltonian (1). The next step now is to calculate the associated eigenvectors of the above eigenvalues $\lambda_k^{\mathrm{I,II}}$. Indeed, we propose to consider two cases: the first case is for $\phi = -1$ and the second one is for $\phi = +1$.

3.1. The first case: $\phi = -1$

Considering $\phi = -1$, the eigenvalues (21) are given by

$$\begin{aligned}\lambda_k^{\mathrm{I}} &= \frac{\hbar\omega k + P(k) + \sqrt{(\hbar\omega k + P(k) - \epsilon)^2 - 4k!\rho^2}}{2},\\ \lambda_k^{\mathrm{II}} &= \frac{\hbar\omega k + P(k) - \sqrt{(\hbar\omega k + P(k) - \epsilon)^2 - 4k!\rho^2}}{2}.\end{aligned} \tag{22}$$

For the sake of simplicity, we can impose $P(k) = 0$ and the eigenvalues $\lambda_k^{\mathrm{I,II}}$ are of the form

$$\lambda_k^{\mathrm{I}} = \frac{\hbar\omega k + \sqrt{(\hbar\omega k - \epsilon)^2 - 4k!\rho^2}}{2}, \qquad \lambda_k^{\mathrm{II}} = \frac{\hbar\omega k - \sqrt{(\hbar\omega k - \epsilon)^2 - 4k!\rho^2}}{2}. \tag{23}$$

The following relations are considered as in [5]:

$$|\hbar\omega k - \epsilon| \geqslant 2\rho\sqrt{k!}, \qquad 2\rho\sqrt{k!} = (\hbar\omega k - \epsilon)\sin\theta_k, \tag{24}$$

and the Hamiltonian matrix given by (19) is written as follows:

$$\begin{aligned}H_k &= \begin{pmatrix} \frac{\epsilon}{2} & \rho\sqrt{k!} \\ -\rho\sqrt{k!} & \hbar\omega k - \frac{\epsilon}{2} \end{pmatrix},\\ &= \begin{pmatrix} \frac{\epsilon}{2} & \frac{1}{2}(\hbar\omega k - \epsilon)\sin\theta_k \\ -\frac{1}{2}(\hbar\omega k - \epsilon)\sin\theta_k & \hbar\omega k - \frac{\epsilon}{2} \end{pmatrix}.\end{aligned} \tag{25}$$

Taking into account the following equation

$$\begin{pmatrix} \frac{\epsilon}{2} & \frac{1}{2}(\hbar\omega k - \epsilon)\sin\theta_k \\ -\frac{1}{2}(\hbar\omega k - \epsilon)\sin\theta_k & \hbar\omega k - \frac{\epsilon}{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \lambda_k^{\mathrm{I,II}} \begin{pmatrix} A \\ B \end{pmatrix}, \tag{26}$$

the associated eigenvectors of $\lambda_k^{\mathrm{I,II}}$ are determined:

$$\left|\psi_k^{\mathrm{I}}\right\rangle = \sin\frac{\theta_k}{2}\left|0, \frac{1}{2}\right\rangle + \cos\frac{\theta_k}{2}\left|k, -\frac{1}{2}\right\rangle, \qquad \text{for} \quad \lambda_k^{\mathrm{I}} = \frac{\hbar\omega k}{2}(1 + \cos\theta_k) - \frac{\epsilon}{2}\cos\theta_k, \tag{27}$$

with $A = \sin\frac{\theta_k}{2}$ and $B = \cos\frac{\theta_k}{2}$.

$$\left|\psi_k^{\mathrm{II}}\right\rangle = \cos\frac{\theta_k}{2}\left|0, \frac{1}{2}\right\rangle + \sin\frac{\theta_k}{2}\left|k, -\frac{1}{2}\right\rangle, \qquad \text{for} \quad \lambda_k^{\mathrm{II}} = \frac{\hbar\omega k}{2}(1 - \cos\theta_k) + \frac{\epsilon}{2}\cos\theta_k, \tag{28}$$

with $A = \cos\frac{\theta_k}{2}$ and $B = \sin\frac{\theta_k}{2}$.

In particular, for $k = 1$, it is easily checked that ψ_k^{I} and ψ_k^{II} become respectively ψ_1^{I} and ψ_1^{II} which were determined in [5].

3.2. The second case: $\phi = +1$

Taking into account $\phi = +1$ and imposing $P(k) = 0$, the eigenvalues (21) become

$$\lambda_k^{\mathrm{I}} = \frac{\hbar\omega k + \sqrt{(\hbar\omega k - \epsilon)^2 + 4k!\rho^2}}{2}, \qquad \lambda_k^{\mathrm{II}} = \frac{\hbar\omega k - \sqrt{(\hbar\omega k - \epsilon)^2 + 4k!\rho^2}}{2}. \tag{29}$$

Accordingly, the relations considered in equation (24) become

$$|\hbar\omega k - \epsilon| \geqslant 2\rho\sqrt{k!}, \qquad 2\rho\sqrt{k!} = (\hbar\omega k - \epsilon)\sinh\theta_k. \tag{30}$$

Following the same method used in the previous case, the eigenvectors associated with above eigenvalues (29) are written as follows:

$$\begin{aligned}
\left|\psi_k^{\rm I}\right\rangle &= \sinh\frac{\theta_k}{2}\left|0,\frac{1}{2}\right\rangle + \cosh\frac{\theta_k}{2}\left|k,-\frac{1}{2}\right\rangle, &\quad \text{for}\quad \lambda_k^{\rm I} &= \frac{\hbar\omega k}{2}(1+\cosh\theta_k) - \frac{\epsilon}{2}\cosh\theta_k,\\
\left|\psi_k^{\rm II}\right\rangle &= \cosh\frac{\theta_k}{2}\left|0,\frac{1}{2}\right\rangle - \sinh\frac{\theta_k}{2}\left|k,-\frac{1}{2}\right\rangle, &\quad \text{for}\quad \lambda_k^{\rm II} &= \frac{\hbar\omega k}{2}(1-\cosh\theta_k) + \frac{\epsilon}{2}\cosh\theta_k,
\end{aligned} \tag{31}$$

For $H \neq H^\dagger$ (i.e., for $\phi = -1$), it can be easily observed that two states given by (27) and (28) are not orthogonal to each other. But one can prove that the states given by equation (31) (i.e., for $\phi = +1$, $H = H^\dagger$) are orthogonal. This property is a consequence of the Hermiticity of H. Hence, one can find the next excited states, by considering the next invariant subspace which is spanned by the vectors $\left|1,\frac{1}{2}\right\rangle$ and $\left|k+1,-\frac{1}{2}\right\rangle$. The eigenvalues and eigenvectors for this doublet can be determined following the same method used previously.

3.3. The excited states

The next step is to generalize the previous results to the invariant subspace which is spanned by the vectors $\left|n,\frac{1}{2}\right\rangle$ and $\left|n+k,-\frac{1}{2}\right\rangle$. Following the same technique used in the previous section and after some algebra, the Hamiltonian matrix for the above doublet is written as

$$H_{n+k} = \begin{pmatrix} \hbar\omega n + P(n) + \frac{\epsilon}{2} & \rho\sqrt{n+1}\cdots\sqrt{n+k} \\ \phi\rho\sqrt{n+1}\cdots\sqrt{n+k} & \hbar\omega(n+k) + P(n+k) - \frac{\epsilon}{2} \end{pmatrix}. \tag{32}$$

For the sake of simplicity, we impose $P(n) = P(n+k) = 0$ and H_{n+k} is of the form

$$H_{n+k} = \begin{pmatrix} \hbar\omega n + \frac{\epsilon}{2} & \rho\sqrt{n+1}\cdots\sqrt{n+k} \\ \phi\rho\sqrt{n+1}\cdots\sqrt{n+k} & \hbar\omega(n+k) - \frac{\epsilon}{2} \end{pmatrix} \tag{33}$$

and its eigenvalues are

$$\begin{aligned}
\lambda_{n+k}^{\rm I} &= \frac{\hbar\omega(2n+k) + \sqrt{(\hbar\omega k - \epsilon)^2 + \phi 4\rho^2(n+1)\cdots(n+k)}}{2},\\
\lambda_{n+k}^{\rm II} &= \frac{\hbar\omega(2n+k) - \sqrt{(\hbar\omega k - \epsilon)^2 + \phi 4\rho^2(n+1)\cdots(n+k)}}{2}.
\end{aligned} \tag{34}$$

In particular, putting $k = 1$ and $\phi = -1$ in (34), the above eigenvalues become the eigenvalues $\lambda_{n+1}^{\rm I,II}$ associated with the operator H given by equation (1). These eigenvalues were determined in [5].

Indeed, putting $2\rho\sqrt{n+1}\cdots\sqrt{n+k} = (\hbar\omega k - \epsilon)\sin\theta_{n+k}$ and $2\rho\sqrt{n+1}\cdots\sqrt{n+k} = (\hbar\omega k - \epsilon)\sinh\theta_{n+k}$ in equation (34) respectively for $\phi = -1$ and for $\phi = +1$, we find the eigenvectors associated with the doublet $\left|n,\frac{1}{2}\right\rangle$ and $\left|n+k,-\frac{1}{2}\right\rangle$.

Let us first consider $\phi = -1$, the eigenvectors associated with this doublet are

$$\begin{aligned}
\left|\psi_{n+k}^{\rm I}\right\rangle &= \sin\frac{\theta_{n+k}}{2}\left|n,\frac{1}{2}\right\rangle + \cos\frac{\theta_{n+k}}{2}\left|n+k,-\frac{1}{2}\right\rangle,\\
&\qquad \text{for}\quad \lambda_{n+k}^{\rm I} = \hbar\omega n + \frac{\hbar\omega k}{2}(1+\cos\theta_{n+k}) - \frac{\epsilon}{2}\cos\theta_{n+k},\\
\left|\psi_{n+k}^{\rm II}\right\rangle &= \cos\frac{\theta_{n+k}}{2}\left|n,\frac{1}{2}\right\rangle + \sin\frac{\theta_{n+k}}{2}\left|n+k,-\frac{1}{2}\right\rangle,\\
&\qquad \text{for}\quad \lambda_{n+k}^{\rm I} = \hbar\omega n + \frac{\hbar\omega k}{2}(1-\cos\theta_{n+k}) + \frac{\epsilon}{2}\cos\theta_{n+k}.
\end{aligned} \tag{35}$$

Finally, considering $\phi = +1$ for equation (34), the eigenvectors for the doublet $\left|n, \frac{1}{2}\right\rangle$ and $\left|n+k, -\frac{1}{2}\right\rangle$ are of the form

$$
\begin{aligned}
\left|\psi^{\mathrm{I}}_{n+k}\right\rangle &= \sinh\frac{\theta_{n+k}}{2}\left|n, \frac{1}{2}\right\rangle + \cosh\frac{\theta_{n+k}}{2}\left|n+k, -\frac{1}{2}\right\rangle, \\
&\qquad \text{for}\quad \lambda^{\mathrm{I}}_{n+k} = \hbar\omega n + \frac{\hbar\omega k}{2}(1+\cosh\theta_{n+k}) - \frac{\epsilon}{2}\cosh\theta_{n+k}, \\
\left|\psi^{\mathrm{II}}_{n+k}\right\rangle &= \cosh\frac{\theta_{n+k}}{2}\left|n, \frac{1}{2}\right\rangle - \sinh\frac{\theta_{n+k}}{2}\left|n+k, -\frac{1}{2}\right\rangle, \\
&\qquad \text{for}\quad \lambda^{\mathrm{I}}_{n+k} = \hbar\omega n + \frac{\hbar\omega k}{2}(1-\cosh\theta_{n+k}) + \frac{\epsilon}{2}\cosh\theta_{n+k}.
\end{aligned}
\tag{36}
$$

Note that all the discussions considered in the previous section are confirmed by these generalized results.

4. Quasi-exactly solvable Hamiltonians

In this section, let us consider an extension of the Jaynes–Cummings Hamiltonian which includes two-photon interaction:

$$H_2 = \frac{\epsilon}{2}\sigma_3 + \hbar\omega a^\dagger a + \rho\left(\sigma_+ a^2 + \sigma_- a^{\dagger 2}\right). \tag{37}$$

The matrix form of the above Hamiltonian leads to

$$H_2 = \begin{pmatrix} \hbar\omega a^\dagger a + \frac{\epsilon}{2} & \rho a^2 \\ \rho(a^\dagger)^2 & \hbar\omega a^\dagger a - \frac{\epsilon}{2} \end{pmatrix}. \tag{38}$$

It is clear that this Hamiltonian H is similar to the one reported in [11] and is also a particular case of the Hamiltonian given by equation (13) (i.e., if $k = 2$, $P(a^\dagger a) = 0$), and one can easily prove its exact solvability. Indeed, if one would like to construct a JC-type Hamiltonian including both one-photon and two-photon interaction, the above Hamiltonian is modified as follows:

$$H_{12} = \begin{pmatrix} \hbar\omega a^\dagger a + \frac{\epsilon}{2} & \rho a^2 + \rho_1 a \\ \rho(a^\dagger)^2 + \hat{\rho}_1 a^\dagger & \hbar\omega a^\dagger a - \frac{\epsilon}{2} \end{pmatrix}. \tag{39}$$

where $\rho, \rho_1, \hat{\rho}_1$ are, *a priori*, arbitrary constants.

Unfortunately, the corresponding operator H_{12} is not any longer exactly solvable. Indeed, it is easy to show that H_{12} does not admit any finite-dimensional invariant vector spaces. Accordingly, it is impossible (to our knowledge) to find its energy spectrum by algebraic methods.

In order to restore, at least in part, a certain algebraic solvability of H_{12}, one can attempt to supplement the Hamiltonian H_{12} with an appropriate interaction term. After some algebra, one can convince oneself that adding an interaction term of the form

$$H_I = \frac{1}{n}\begin{pmatrix} 0 & \rho_1 a a^\dagger a \\ \hat{\rho}_1 a^\dagger a a^\dagger & 0 \end{pmatrix} \tag{40}$$

leads to a new Hamiltonian $H_T = H_{12} + H_I$ which is quasi-exactly solvable, as we will now demonstrate.

Assuming n to be an integer and redefining $c \equiv -\frac{\rho_1}{n}$, $\hat{c} \equiv -\frac{\hat{\rho}_1}{n}$, the operator H_T is of the form

$$H_T = \begin{pmatrix} \hbar\omega a^\dagger a + \frac{\epsilon}{2} & \rho a^2 + ca(a^\dagger a - n) \\ \phi\rho(a^\dagger)^2 + \hat{c}(a^\dagger a - n)a^\dagger & \hbar\omega a^\dagger a - \frac{\epsilon}{2} \end{pmatrix}, \tag{41}$$

where $a^\dagger$ and a are respectively the usual creation and annihilation operators and ϵ is chosen as previously according to $\epsilon = 2\mu B_0$.

The main idea now is to reveal that the above operator H_T is quasi-exactly solvable (QES). In this purpose, we construct a finite-dimensional vector space which is invariant under the action of H_T. Let us now apply the Hamiltonian H to the states $\binom{|N\rangle}{0}$ and $\binom{0}{|M\rangle}$ with $N, M \in \mathbb{N}$ as follows:

$$H_T\begin{pmatrix}|N\rangle\\|M\rangle\end{pmatrix} = \begin{pmatrix}\left(\hbar\omega N + \frac{\epsilon}{2}\right)|N\rangle + \rho\sqrt{M(M-1)}|M-2\rangle + c\sqrt{M}(M-n)|M-1\rangle\\ \phi\rho\sqrt{(N+1)(N+2)}|N+2\rangle + \hat{c}\sqrt{N+1}(N+1-n)|N+1\rangle + \left(\hbar\omega M - \frac{\epsilon}{2}\right)|M\rangle\end{pmatrix}. \tag{42}$$

In order to be in agreement with the invariance of the two vectors states $\binom{|N\rangle}{0}$ and $\binom{0}{|M\rangle}$ under the action of the Hamiltonian H_T, we have to impose the value of the integer n according to $n = M = N+2$ (i.e., $N = M-2$). Taking into account the above fixed value of n, we obtain

$$H_T\begin{pmatrix}|N\rangle\\|M\rangle\end{pmatrix} = \begin{pmatrix}\left[\left(\hbar\omega N + \frac{\epsilon}{2}\right) + \rho\sqrt{(N+2)(N+1)}\right]|N\rangle\\ \left[\hbar\omega(N+2) - \frac{\epsilon}{2} + \phi\rho\sqrt{(N+1)(N+2)}\right]|N+2\rangle - \hat{c}\sqrt{N+1}|N+1\rangle\end{pmatrix}. \tag{43}$$

Finally, the Hamiltonian H_T is of the new form

$$H_T = \begin{pmatrix}\hbar\omega a^\dagger a + \frac{\epsilon}{2} & \rho a^2 + ca(a^\dagger a - (N+2))\\ \pm\rho(a^\dagger)^2 + \hat{c}(a^\dagger a - (N+2))a^\dagger & \hbar\omega a^\dagger a - \frac{\epsilon}{2}\end{pmatrix}. \tag{44}$$

Obviously, from equation (43), one can easily check that the Hamiltonian H_T preserves the finite-dimensional vector space, namely,

$$\mathcal{V}_n = \text{span}\left\{\begin{pmatrix}|j\rangle\\0\end{pmatrix}, \begin{pmatrix}0\\|k\rangle\end{pmatrix}, j = N, \ldots, 0; k = N+2, \ldots, 0\right\}, \tag{45}$$

and n is replaced by $N+2$. From this, we conclude that the Hamiltonian H_T is quasi-exactly solvable. Hence, the terms of perturbation added to H_{12} have broken its non-solvability.

Note that it is also easy to reveal the quasi-exact solvability of the operator expressed in equation (41) by considering the matrix Hamiltonian equation (41) in terms of differential expressions and variable x. Replacing the operators $a^\dagger$ and a respectively by their differential expressions given by equation (3), performing the standard gauge transformation as

$$\tilde{H}_T = \exp\left(\frac{\omega x^2}{2}\right) H_T \exp\left(-\frac{\omega x^2}{2}\right), \tag{46}$$

and thus, after some algebra, we obtain a matrix Hamiltonian which preserves the finite-dimensional vector space of the form $\mathcal{V}_k = (P_k(x), P_{k+2}(x))^t$ with $k \in \mathbb{N}$ and $n = k+2$ (i.e., n which is expressed in equation (41)). This operator $\tilde{H}_T$ (therefore H_T) is quasi-exactly solvable because it is expressed in terms of the integer n which is fixed as $n = k+2$.

5. Spectral properties

In this section, we would like to emphasize a few properties of the spectrum of the Hamiltonian discussed above. First, we stress that for given k the JC model admits k levels which are ρ-independent and which are not involved in the list given above. They are of the form

$$\psi_j = \begin{pmatrix}\vec{0}\\|j\rangle\end{pmatrix}, \qquad 0 \leqslant j \leqslant k-1,$$

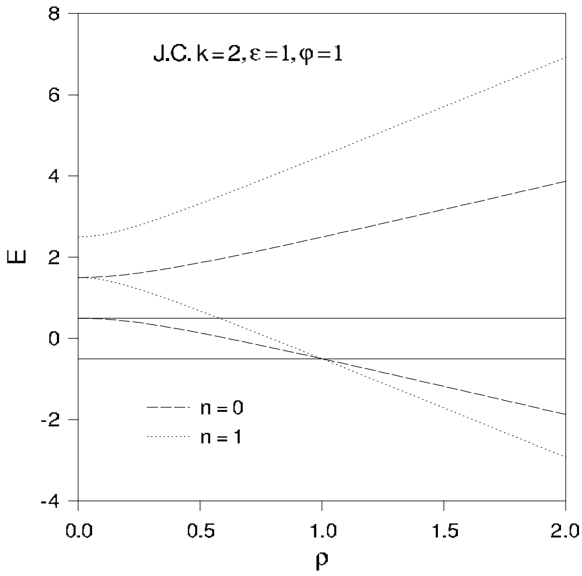

Figure 1. The first few energy levels in the $k = 2$ JC Hamiltonian for $\epsilon = 1$ and $\phi = 1$.

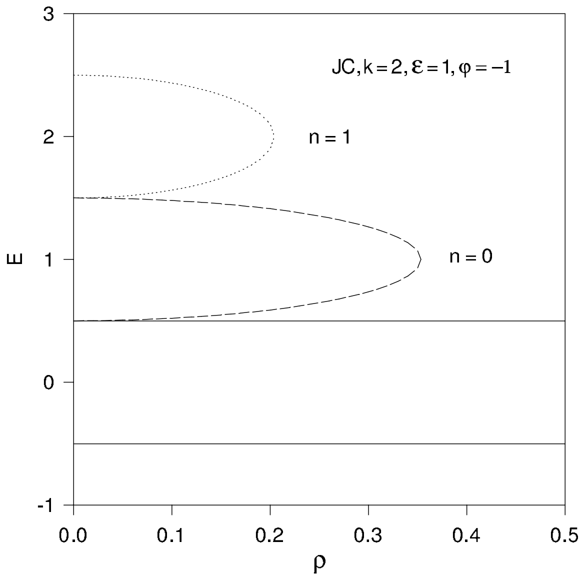

Figure 2. The first few energy levels in the $k = 2$ JC Hamiltonian for $\epsilon = 1$ and $\phi = -1$.

where $\vec{0}$ denotes the null vector of the Hilbert space. The corresponding eigenvalue is $E_j = j - \frac{\epsilon}{2}$.

The spectrum of the JC model (and of its generalizations for $k > 1$) varies considerably with the parameter ρ. In figure 1, we show the evolution of six levels in the $k = 2, \phi = 1$ case. They correspond to the two ρ-independent eigenstates and the ones with $n = 0, 1$ in equation (34). In figure 1 and in the following, we assume $\epsilon = 1$ for simplicity but the features pointed out below remain similar for $\epsilon \neq 1$. The same levels corresponding to the non-Hermitian case $\phi = -1$ are reported in figure 2. The contrast with figure 1 is obvious.

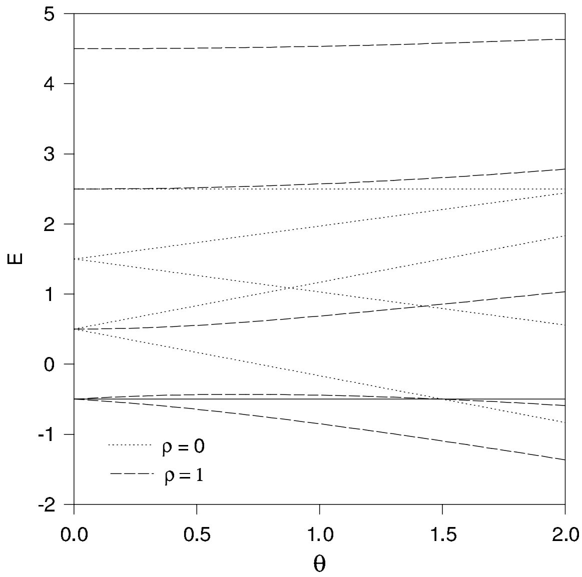

Figure 3. The first few energy levels in the QES deformed $k = 2$ JC Hamiltonian as a function of the parameter θ, the energy level $E = -1/2$ (solid line) is independent of ρ.

Couples of eigenvalues regularly disappear at finite values of the coupling constants ρ. So that, at finite ρ only a finite number of real eigenvalues subsist, the other being real. In this respect, the Hamiltonian is like a quasi-exactly solvable operator.

The energy levels displayed in figure 1 correspond to the six lowest ones in the limit $\rho = 0$. The figure clearly shows that they mix relatively quickly for increasing ρ and that, for instance, eigenvectors involving two or more quanta become the ground state for $\rho \sim 1$.

We have studied the evolution of the spectrum when the QES extension of the model, $H_{12} = \rho a^2 + \theta a\left(1 - \frac{1}{N+2}a^\dagger a\right)$ namely characterized by the new coupling constant θ, is progressively switched on. Note that the vector $\psi_0 = (\vec{0}, |0\rangle)^t$ is an eigenvector with $E = -\epsilon/2$, irrespectively of ρ, θ.

In the case $\rho = 0$, $N = 1$, the effect of the new term on the eigenvalues under consideration leads to

$$E = -\tfrac{1}{2}, \tfrac{1}{6}(3 \pm 4\theta), \tfrac{1}{6}(9 \pm 2\sqrt{2})\theta, \tfrac{5}{2}.$$

These levels are indicated in figure 3 by the dotted lines and it is clearly seen that they also lead to numerous level crossing.

The evolution of the eigenvalues corresponding to the case $\rho = 1$ is displayed by the dashed lines in figure 3, supplemented by the black line $E = -1/2$ which is present irrespectively of ρ. The figure clearly shows that the occurrence of the new term induced only one level mixing, namely two levels cross at $E = -1/2$ for $\theta = 1.5$. For larger values of ρ, e.g. $\rho = 2$, the analysis reveals that the algebraic eigenvalues depend only weakly on θ.

6. Series expansion and recurrence relations

Here, we would like to present another aspect of the QES Hamiltonian presented in the previous section. Following the ideas of [15], we will construct the solution for energy E

under the form of a formal series in the basic vector whose coefficients are polynomials in E. More precisely, we write the solution of the equation

$$H_T\psi = E\psi, \tag{47}$$

in the form

$$\psi = \begin{pmatrix} \sum_{j=0}^{\infty} p_j(E)|j\rangle \\ \sum_{j=-2}^{\infty} q_j(E)|j+2\rangle \end{pmatrix} \tag{48}$$

where H_T is given by equation (41). After some algebra it can be seen that the polynomials $p_j(E), q_j(E)$ obey the following recurrence relations:

$$A_{j+1}P_{j+1} + B_j P_j = 0, \tag{49}$$

where

$$A_{j+1} = \begin{pmatrix} \rho\sqrt{(j+2)(j+3)} & -\left(E-(j+1)-\frac{\epsilon}{2}\right) \\ 0 & \hat{c}(j+2-n)\sqrt{j+2} \end{pmatrix},$$
$$B_j = \begin{pmatrix} c(j+2-n)\sqrt{j+2} & 0 \\ -\left(E-(j+2)+\frac{\epsilon}{2}\right) & \rho\sqrt{(j+1)(j+2)} \end{pmatrix}, \tag{50}$$
$$P_j = \begin{pmatrix} q_j \\ p_j \end{pmatrix}, \qquad j = -2, -1, 0, 1, \ldots.$$

These equations have to be solved with the initial conditions

$$q_{-2} = 0, q_{-1} = \mathcal{N}, \tag{51}$$

with $\mathcal{N}$ fixing the normalization of the solution. Then, the solution for q_j turns out to be a polynomial of degree E^{2j}. The quasi-exact solvability of the system leads to the fact that A_{n-1} is not invertible and that p_{n-1} can be chosen arbitrarily. With the choice $p_{n-1} = 0$ it turns out that all polynomials p_j, q_j with $j \geqslant n-2$ are proportional to $q_{n-3}(E)$. As a consequence for fixed n and for the values of E such that $q_{n-3}(E) = 0$, the series above is truncated and the set of algebraic eigenvectors are recovered. We would like to stress that series considered in this section are built with the basis vector of the harmonic oscillator and not on monomials in x contrasting with the construction of [15]. In the case of standard QES equations [15], it appears three terms' recurrence relations which lead to sets of orthogonal relation. In the case of systems of QES equations addressed in [16], the recurrence relation is also three terms but the situation here is quite different. Actually, it is to our knowledge, an open question to know whether the set of polynomials $(p_j(E), q_j(E))$ is somehow orthogonal as it is the case for standard scalar equation.

7. Hidden algebraic structures

As pointed out in the previous sections, the different Hamiltonians studied here possess the property that their spectrum can be (partly or fully) computed. This property is deeply related to the fact that the corresponding operators are elements of the enveloping algebra of particular graded algebra in an appropriate finite-dimensional representation. The classification of linear operators preserving the vector spaces $\mathcal{V}(m,n) = (P_m(x), P_n(x))^t$ was reported in [10]. It is shown that these operators are the elements of the enveloping algebra of some nonlinear graded algebra depending essentially on $|m-n|$. Note that, in the present context, the difference $|m-n|$ is nothing else but the parameter called k in section 3. The cases $k = 1$ and $k = 2$ are special because the underlying algebra is indeed a graded Lie algebra. In the case $k = 1$, related to the conventional JC model, the Hamiltonian is an element of the enveloping algebra of $osp(2, 2)$, in the representation constructed in [9]. The generators involved in this relation do not depend explicitly on n, i.e. on the dimension of the representation, explaining that the Hamiltonian is exactly solvable. Finally, in the case $k = 2$, the Hamiltonian is an element of the graded Lie algebra $q(2)$, as shown in [17, 18]. This algebra possesses an $sl(2) \times U(1)$ bosonic subalgebra and six fermionic operators split into three triplets of the $sl(2)$ subalgebra. In the case of the JC model corresponding to $k = 2$, the Hamiltonian is independent of the dimension of the representation n and the model is exactly solvable. For the modified model of section 4, the supplementary interaction term H_I defined in (40) depends on n and the operator admits only the vector space $\mathcal{V}_n$ as finite-dimensional invariant vector space.

8. Conclusions

In this paper, we have considered several extensions of the Jaynes–Cummings (JC) model by adding to its original Hamiltonian the polynomial $P(a^\dagger a)$ of degree $d \geqslant 2$ and $\phi = \pm 1$ in the non-diagonal interaction term. In fact, considering the sign $\phi = -1$, these extended Hamiltonians are non-Hermitian and not PT invariant but they satisfy the pseudo-Hermiticity with respect to different operators P and σ_3. This new property reveals the reality of the energy spectrum which has been constructed algebraically. The Hamiltonians become Hermitian when one considers $\phi = +1$. Accordingly, these Hamiltonians are completely solvable as it has been pointed out by using standard QES techniques.

Several well-known properties of Hermitian Hamiltonians are not kept with the pseudo-Hermitian one. One example is the orthogonality of eigenstates. We have shown in this paper that the eigenstates corresponding to the doublets $\left|0, \frac{1}{2}\right\rangle$ and $\left|k, -\frac{1}{2}\right\rangle$ are not orthogonal to each other, however, they are orthogonal to all eigenstates corresponding to other doublets. The eigenstates of any particular doublet are orthogonal to each other only if $\theta_m = m\pi$ (i.e., with $m = 0, 1, 2, \ldots, k, \ldots, n + k$). This implies $\rho = 0$ because it depends on $\sin\theta_m$. In fact, as the energy eigenvalues are entirely real, it is impossible to have all eigenstates orthogonal to each other. This is explained by the unbroken symmetry of the operator $P\sigma_3$. But for complex energy eigenvalues the orthonormality condition is satisfied by all the associated eigenstates. All these discussions are the result of the scalar product applied to those eigenstates.

We have constructed a JC-type Hamiltonian describing both one- and two-photon interactions in terms of quasi-exactly solvable operators. This involves a very specific interaction term of degree 1 in the creation and annihilation operators which can be seen as a perturbation of more conventional p-photon interaction terms. Several properties of this new family of QES operators have been presented. Namely, (i) they can be written in terms of the generators of the graded Lie algebra $osp(2, 2)$ in a suitable representation and (ii) when expressed as series, the formal solutions of $H_T\psi = E\psi$ lead to a different type of recurrence relation between the different terms of the series.

References

[1] Bender C M and Boettcher S 1998 *Phys. Rev. Lett.* **80** 5243
Bender C M, Boettcher S and Meisinger P N 1999 *J. Math. Phys.* **40** 2210

[2] Bender C M and Boettcher S 2002 *Phys. Rev. Lett.* **89** 270401

[3] Bender C M *Preprint* hep-th/0501052

[4] Mostafazadeh A 2002 *J. Math. Phys.* **43** 205
Mostafazadeh A 2002 *J. Math. Phys.* **43** 2814
Mostafazadeh A 2002 *J. Math. Phys.* **43** 3944
Mostafazadeh A and Batal A 2004 *J. Phys. A: Math. Gen.* **37** 11645

[5] Mandal B P 2005 *Mod. Phys. Lett* A **20** 655 (*Preprint* hep-th/0412160)
[6] Ghosh P K 2005 *J. Phys. A: Math. Gen.* **38** 7313 (*Preprint* quant-ph/0501087)
[7] Turbiner A 1988 *Commun. Math. Phys.* **119** 467
[8] Ushveridze A G 1995 *Quasi Exactly Solvable Models in Quantum Mechanics* (Beograd: Institute of Physics)
[9] Shifman M and Turbiner A 1989 *Commun. Math. Phys.* **126** 347
[10] Brihaye Y and Kosinski P 1994 *J. Math. Phys.* **35** 3089
[11] Gerry C C 1988 *Phys. Rev.* A **37** 2683
[12] Deb B and Ray D S 1993 *Phys. Rev.* A **48** 3191
[13] Knight P L and Radmore P M 1982 *Phys. Rev.* A **26** 676
[14] Debergh N and Klimov A B 2001 *J. Mod. Phys.* **16** 4057
[15] Bender C M and Dunne G V 1996 *J. Math. Phys.* **37** 6
[16] Brihaye Y, Ndimubandi J and Mandal B P 2006 QES polynomials, invariant spaces and polynomials recursion *Preprint* math-ph/0601004 (*Mod. Phys. Lett.* A at press)
[17] Debergh N and Van der Jeugt J 2001 *J. Phys. A: Math. Gen.* **34** 8119
[18] Brihaye Y and Hartmann B 2003 *Phys. Lett.* A **306** 291